SOME REMARKS ON WILLMORE SURFACES EMBEDDED IN $\mathbb{R}^3$

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Abstract. Let $f : \mathbb{C} \to \mathbb{R}^3$ be complete Willmore immersion with $\int_\Sigma |A_f|^2 < +\infty$. We will show that if f is the limit of an embedded surface sequence, then f is a plane. As an application, we prove that if Σ_k is a sequence of closed Willmore surface embedded in $\mathbb{R}^3$ with $W(\Sigma_k) < C$, and if the conformal class of Σ_k converges in the moduli space, then we can find a Möbius transformation σ_k, such that a subsequence of $\sigma_k(\Sigma_k)$ converges smoothly.

1. Introduction

Let $f : \Sigma \to \mathbb{R}^3$ be an embedding. We define the first and second fundamental form of f as follows:

$$g = g_{ij}dx^i \otimes dx^j = df \otimes df, \quad \textit{and} \quad A = A_{ij}dx^i \otimes dx^j = -df \otimes dn.$$

Let $H = g^{ij}A_{ij}$ be the mean curvature, and K be the Causs curvature. It is well-known that

$$\vec{H} = Hn = \Delta_g f. \tag{1.1}$$

We say f is minimal, if $H = 0$, and Willmore if H satisfies the equation:

$$\Delta_g H + \frac{1}{2}(|H|^2 - 4K)H = 0. \tag{1.2}$$

Note that (1.2) is the Euler-Langrange equation of Willmore functional [19]:

$$W(f) = \frac{1}{4}\int |H|^2 d\mu_g.$$

Now, we let f be an embedding from $\mathbb{C}$ into $\mathbb{R}^3$. We assume f is complete, noncompact, with $\int_{\mathbb{C}} |A|^2 < +\infty$. It is well-known that when f is minimal, f must be a plane [15]. In this paper, we will show that such a result is also true when f is Willmore:

Theorem 1.1. *Let $f : \mathbb{C} \to \mathbb{R}^3$ be an complete Willmore embedding. If $\int_{\mathbb{C}} |A|^2 < +\infty$, then $f(\mathbb{C})$ is a plane.*

Remark 1.2. *In [4], Chen and Lamm has proved that any Willmore graph over $\mathbb{R}^2$ in $\mathbb{R}^3$ must be plane, whenever it has finite $\|A\|_{L^2}$.*

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Luo and Sun proved that if the Willmore functional of the Willmore graph is finite, then $\|A\|_{L^2}$ is finite [13]. *However, this is not true for an embedded Willmore surface. For example, helicoids are embedded minimal surfaces ($W = 0$), but have infinite $\|A\|_{L^2}$.*

Next, we will show that Theorem 1.1 still holds if we replace 'embedding' with 'the limit of an embedding sequence':

Theorem 1.3. *Let $f : \mathbb{C} \to \mathbb{R}^3$ be a conformal complete Willmore immersion with $\int_{\mathbb{C}} |A|^2 < +\infty$. If there exist $R_k \to +\infty$ and embedding $\phi_k : D_{R_k} \to \mathbb{R}^3$, such that f_k converges to f in $C^1(D_R)$ for any R, then $f(\mathbb{C})$ is a plane.*

As an application, we will prove the following:

Theorem 1.4. *Let Σ_k be a sequence of closed Willmore surface embedded in $\mathbb{R}^3$. We assume the genus is fixed and $W(\Sigma_k) < C$. If the conformal class of Σ_k is contained in a compact subset of the moduli space, then we can find Möbius transformation σ_k, such that $\sigma_k(\Sigma_k)$ converges smoothly.*

Remark 1.5. *Let Σ_k be a Willmore surface immersed in $\mathbb{R}^3$ and $\mathbb{R}^4$. Bernard and Riviére* [1] *proved that if*

$$W(\Sigma_k) < \min\{8\pi, \omega_g^n\} - \delta,$$

modulo the action of the Möbius group, $\{\Sigma_k\}$ is compact. By results in [8] *(see also* [17]*), when $W(\Sigma_k) < \min\{8\pi, \omega_g^n\} - \delta$, the conformal class of Σ_k must be compact in the moduli space. Moreover, by Li-Yau's inequality* [11]*, Σ_k is an embedding when $W(\Sigma_k) < 8\pi$.*

When f has no branches at ∞, Theorem 1.1 and Theorem 1.3 can be deduced directly from the removability of singularity [9] and the classification of Willmore sphere in S^3 [2]. In fact, the results in [2] imply the following:

Lemma 1.6. *Let $f : S^2 \to \mathbb{R}^3$ be a Willmore immersion. If f has no transversal self-intersectiones, then f is an embedding and $f(S^2)$ is a round sphere.*

Then, to get Theorem 1.1 and 1.3, we only need to prove f has no branches at ∞. For this sake, we will prove the following:

Lemma 1.7. *Let $f : \mathbb{C} \setminus D_R \to \mathbb{R}^3$ be a smooth conformal complete embedding with*

$$\|A\|_{L^2(\mathbb{C}\setminus D_R)} < +\infty, \quad \varlimsup_{|z|\to+\infty} |f(z)| \cdot |A(z)| < +\infty.$$

Then

$$\theta^2\big(f(\mu_g \llcorner \mathbb{C} \setminus D_R), \infty\big) = 1.$$

Lemma 1.8. *Let $f : \mathbb{C} \setminus D_R \to \mathbb{R}^3$ be a smooth conformal complete immersion with*

$$\|A\|_{L^2(\mathbb{C}\setminus D_R)} < +\infty, \quad \varlimsup_{|z|\to+\infty} |f(z)| \cdot |A(z)| < +\infty.$$

If there exists embedding $\phi_k : D_{R_k} \setminus D_R \to \mathbb{R}^3$, which converges to f_0 in $C^2(D_{R'} \setminus D_R)$ for any $R' > R$, then

$$\theta^2\big(f(\mu_g \llcorner \mathbb{C} \setminus D_R), \infty\big) = 1.$$

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2. Complete Willmore embedding of $\mathbb{C}$ in $\mathbb{R}^3$ with $\int |A|^2 < +\infty$

In this section, we will prove Lemma 1.7, and use it to prove Theorem 1.1.

2.1. **The proof of Lemma 1.7.** By Theorem 4.2.1 in [14], we may assume

$$g = e^{2u} g_{euc}, \quad with \quad u = m \log |z| + \omega, \tag{2.1}$$

where m is a nonnegative integer and $\lim_{z\to\infty} \omega(z)$ exists. Moreover, we have

$$\lim_{|z|\to+\infty} \frac{|f|}{|z|^{m+1}} = \frac{e^{w(\infty)}}{m+1}. \tag{2.2}$$

Also by Theorem 4.2.1 in [14], we can obtain

$$\theta^2\big(f(\mu_g \llcorner \mathbb{C} \setminus D_R), \infty\big) = m + 1. \tag{2.3}$$

Let $f_k(z) = \frac{f(r_k z)}{r_k^{m+1}}$, where $r_k \to +\infty$. Let H_k, A_k be the mean curvature and the second fundamental form of f_k respectively. By (1.1),

$$\Delta f_k = \frac{1}{2} \vec{H}_k |\nabla f_k|^2.$$

Since $|\nabla f_k| = \sqrt{2}|z|^m e^{\omega(r_k z)}$, we have

$$\|\Delta f_k\|_{L^2(D_r \setminus D_{\frac{1}{r}})} \leq C(r) W(f_k, D_r \setminus D_{\frac{1}{r}}), \text{ and } \lim_{k\to+\infty} \Big\| |z|^{-m}|\nabla f_k| - \sqrt{2} e^{\omega(\infty)} \Big\|_{C^0(D_r\setminus D_{\frac{1}{r}})} = 0.$$

Noting that $W(f_k, D_r \setminus D_{\frac{1}{r}}) \leq W(f)$ and $f_k(1) \to \frac{e^{\omega(\infty)}}{m+1}$, we get

$$\|\Delta f_k\|_{L^2(D_r \setminus D_{\frac{1}{r}})} + \|f_k\|_{W^{1,2}(D_r \setminus D_{\frac{1}{r}})} < C(r).$$

Applying elliptic estimates, we have $\|f_k\|_{W^{2,2}(D_r \setminus D_{\frac{1}{r}})} < C(r)$. Thus we may assume f_k converges to f_0 weakly in $W^{2,2}(D_r \setminus D_{\frac{1}{r}})$. Then we may assume $df_k \otimes df_k$ converges to $df_0 \otimes df_0$ in $L^q(D_r \setminus D_{\frac{1}{r}})$ for any $q > 0$. Noting that

$$df_k \otimes df_k = |z|^{2m} e^{2\omega(r_k z)} g_{euc},$$

we get

$$df_0 \otimes df_0 = |z|^{2m} e^{2\omega(\infty)} g_{euc}.$$

Let A_0 be the second fundamental form of f_0. Obviously,

$$\int_{D_r \setminus D_{\frac{1}{r}}} |A_0|^2 \leq \lim_{k\to+\infty} \int_{\mathbb{C}\setminus D_{\frac{r_k}{r}}} |A|^2 = 0,$$

then $\int_{\mathbb{C}} |A_0|^2 = 0$ and the imagine f_0 is in a plane. Without loss of generality, we may assume $w(\infty) = 0$ and $f_0 = (z^{m+1}, c)$.

Next, we prove $m = 0$ by contradiction. Assume $m > 0$. By (2.2), when $z \in D_r \setminus D_{\frac{1}{r}}$,

$$|A_k(z)| = r_k^{m+1} |A(r_k z)| = \frac{r_k^{m+1}}{|f(r_k z)|} |f(r_k z)||A(r_k z)| < C(r).$$

Then $\|\Delta f_k\|_{L^\infty(D_r \setminus D_{\frac{1}{r}})} < C$ and f_k converges in fact in $C^1(D_r \setminus D_{\frac{1}{r}})$.

If we set $f_k = (\varphi_k, f_k^3)$, then

$$\varphi_k \to z^{m+1}, \quad f_k^3 \to c \quad in \quad C^1(D_r \setminus D_{\frac{1}{r}}).$$

Let

$$\Sigma_k = f_k(\mathbb{C} \setminus D_{\frac{R}{r_k}}) \cap \left((D_4 \setminus D_{\frac{1}{4}}) \times \mathbb{R} \right).$$

and

$$F_k(x^1, x^2, x^3) = \sqrt{(x^1)^2 + (x^2)^2}.$$

Then F_k is C^1-smooth on Σ_k with no critical points when k is sufficiently large.

Obviously, $\{y \in \Sigma_k : F_k(y) = 1\}$ consists of compact C^1 smooth 1-dimensional manifolds. Since $\varphi_k \to z^{m+1}$ and f_k is an embedding, $\{z : F_k = 1\}$ has at least 2 components. Let $\{F_k = 1\} = \Gamma_1 \cup \Gamma_2 \cdots \cup \Gamma_{m'}$, where Γ_i are components of $\{F_k = 1\}$ and $m' \geq 2$. Let $\phi(\cdot, t)$ be the flow generated by $\nabla F_k / |\nabla F_k|$ and put $\Omega_i = \phi(\Gamma_i, [-\frac{1}{2}, 2])$. Then

$$\bigcup_i \Omega_i = \{2 \geq F_k \geq \frac{1}{2}\}, \quad and \quad \Omega_i \cap \Omega_j = \emptyset.$$

That is to say that $\{2 \geq F_k \geq \frac{1}{2}\}$ has at least 2 components, and on each component Ω_i, we can find y_i such that $F_k(y_i) = 1$.

Let $y_i = f_k(z_i)$. Recall that for any fixed small ϵ, when k is sufficiently large, we have

$$-\epsilon \leq |\varphi_k(z_i)| - |z_i|^{m+1} < \epsilon, \quad i = 1, 2.$$

We may assume z_1, $z_2 \in D_{1+\epsilon'} \setminus D_{1-\epsilon'}$ such that

$$\epsilon' \ll \frac{1}{2}, \quad and \quad D_{1+\epsilon'} \setminus D_{1-\epsilon'} \subset \{z : \frac{3}{2} \geq |\varphi_k(z)| \geq \frac{3}{4}\}.$$

Take a curve γ such that $\gamma([0,1]) \subset D_{1+\epsilon'} \setminus D_{1-\epsilon'}$, and $\gamma(0) = z_1$, $\gamma(1) = z_2$. Then

$$f_k(\gamma(0)) = y_1, \quad f_k(\gamma(1)) = y_2, \quad and \quad f_k(\gamma) \subset \bigcup_i \Omega_i.$$

It is a contradiction to the fact that Ω_1 and Ω_2 are different components. $\square$

2.2. **The proof of Theorem 1.1.** By a result of Huber[7], we may assume f to be conformal. Without loss of generality, we assume $f(0) = 0$. We may assume $\|A\|_{L^2(\mathbb{C} \setminus B_R)} < \epsilon$. Then by Theorem 2.10 in [9],

$$r\|A\|_{L^\infty(B_{2r} \setminus B_r(0))} < C\|A\|_{L^2(B_{4r} \setminus B_{\frac{r}{2}}(0))}$$

whenever $r > 2R$.

Let Σ be the image of embedding $f : \mathbb{C} \to \mathbb{R}^3$. We deduce from Lemma 1.7 that

$$\lim_{R \to +\infty} \frac{\mu_\Sigma(B_R)}{\pi R^2} = 1.$$

Let $y_0 \notin \Sigma$ and $I(y) = \frac{y - y_0}{|y - y_0|^2}$. By Lemma 4.3 in [10], $I(\Sigma)$ can be extended to a smooth closed surface. It is easy to check that $I(\Sigma)$ is an embedded Willmore sphere. By Lemma 1.6, $I(\Sigma)$ must be a round sphere, which implies that Σ is a plane. Then we get Theorem 1.1.

3. Compactness of a Willmore embedding sequence in $\mathbb{R}^3$

In this section, we first prove Lemma 1.8, then prove Theorem 1.4. Since the proof of Theorem 1.3 is very similar to Theorem 1.1, we omit it.

3.1. **The proof of Lemma 1.8.** We assume

$$g = e^{2u} g_{euc}, \quad with \quad u = m \log |z| + \omega,$$

where m is a nonnegative integer and $\lim_{z\to\infty} \omega(z)$ exists. Similar to the proof of Lemma 1.7, we let $f_{0,n}(z) = \frac{f(r_n z)}{r_n^{m+1}}$, where $r_n \to +\infty$. we may assume $f_{0,n}$ converges to (z^{m+1}, c) in $C^1(D_{\frac{1}{r}} \setminus D_r)$.

Recall that ϕ_k converges to f in C^1. Then, we can find k_n, such that $\phi_{k_n}(r_n z)$ converges to (z^{m+1}, c). Then using the the arguments similar as we prove Lemma 1.7, we can finish the proof of Lemma 1.8.

3.2. **The proof of Corollary 1.4.** Let f_k be conformal immersion of (Σ, h_k) into $\mathbb{R}^3$, where h_k is a smooth metric with constant curvature. When the genus of Σ is 1, we assume $\mu(h_k) = 1$. Since the conformal structure induced by h_k converges in the moduli space, we may assume h_k converges smoothly to h_0. By results in [8], we may find Möbius transformation σ_k and a finite set $\mathcal{S}$, such that $\sigma(f_k) = 1$ and $\sigma_k(f_k)$ converges in $W^{2,2}_{loc}(\Sigma \setminus \mathcal{S}, h_0)$, where

$$\mathcal{S} = \{p \in \Sigma : \lim_{r\to 0} \varliminf_{k\to+\infty} \int_{B_r^{h_0}(z)} |A_{f_k}|^2 \geq 8\pi\}.$$

Let f_0 be the limit, which is a branched $W^{2,2}$-conformal immersion. Thus f_0 is continuous on Σ.

The following theorems will be useful, see [5], [16] for proofs respectively.

Theorem 3.1. *Let g_k, g be smooth Riemannian metrics on a surface M, such that $g_k \to g$ in $C^{s,\alpha}(M)$, where $s \in N$, $\alpha \in (0,1)$. Then for each $p \in M$ there exist neighborhoods U_k, U and smooth conformal diffeomorphisms $\varphi_k : D \to U_k$, such that $\vartheta_k \to \vartheta$ in $C^{s+1,\alpha}(\overline{D}, M)$.*

Theorem 3.2. *Let $f : D \to \mathbb{R}^n$ be a conformal immersion with $g_f = e^{2u} g_{euc}$. Assume f is Willmore. Then there exists an $\epsilon_0 > 0$ and a $\lambda > 0$, such that if*

$$\int_D |A|^2 dx < \epsilon_0, \quad and \quad |u| < \lambda,$$

then

$$\|\nabla^k n\|_{L^\infty(D_r)} \leq C(\epsilon_0, \lambda, r) \|A\|_{L^2(D)},$$

where $\nabla = (\frac{\partial}{\partial x^1}, \frac{\partial}{\partial x^2})$.

For simplicity, choose $\epsilon_0 < 8\pi - \delta$.

Lemma 3.3. *Let $f : D \setminus \{0\} \to \mathbb{R}^3$ be a conformal Willmore immersion with*

$$\mu_f(D) + \|A\|_{L^2(\mathbb{C}\setminus D_R)} < +\infty.$$

If there exist $r_k \to 0$ and embedding $f_k : D \setminus D_{r_k} \to \mathbb{R}^3$, which converges to f in $C^1(D\setminus D_r)$ for any $r < 1$, then for any sufficiently small r

$$\theta^2\big(f(\mu_f \llcorner D_r), 0\big) = 1.$$

Proof. Set $g = e^{2u} g_{euc}$. Using Proposition 4.1 in [8], $f \in W^{2,2}(D)$, and

$$u = m \log |z| + \omega(z),$$

where m is positive integer and $\omega \in C^0(D)$. Moreover, we have

$$\lim_{|z|\to 0} \frac{|f(z) - f(0)|}{|z|^{m+1}} = \frac{e^{\omega(0)}}{m+1},$$

and

$$\theta^2\big(f(\mu_g \llcorner D_r), 0\big) = m+1.$$

Without loss of generality, we assume $f(0) = 0$.

Set $\tilde{f} = \frac{f}{|f|^2}$, and $\tilde{g} = d\tilde{f} \otimes d\tilde{f}$. Then $\tilde{g} = \frac{g}{|f|^4}$. Let $A^0 = A - \frac{1}{2} H g$, which is the traceless part of A. It is well-known that

$$\int_D |A^0|^2 d\mu_g = \int_D |\tilde{A}^0|^2 d\mu_{\tilde{g}}.$$

Put $\tilde{g} = e^{2\tilde{u}} g_{euc}$. We have $\tilde{u} = u - \log |f|^2$. By Gauss curvature equation

$$-\Delta \tilde{u} = \tilde{K} e^{2\tilde{u}},$$

we get

$$\begin{aligned}
\int_{D_\delta \setminus D_r} \tilde{K} e^{2\tilde{u}} &= -\int_{\partial D_\delta} \frac{\partial \tilde{u}}{\partial r} + \int_{\partial D_r} \frac{\partial \tilde{u}}{\partial r} \\
&= -\int_{\partial D_\delta} \frac{\partial \tilde{u}}{\partial r} + \int_{\partial D_r} \frac{\partial u}{\partial r} - \int_{\partial D_r} \frac{2 f_r f}{|f|^2} \\
&= \int_{D_\delta \setminus D_r} K e^{2u} - \int_{\partial D_r} \frac{2 f_r f}{|f|^2}.
\end{aligned}$$

Since

$$\left| \int_{\partial D_r} 2 \frac{f_r f}{|f|^2} \right| \leq 2 \int_{\partial D_r} \frac{|f_r|}{|f|} = 2 \int_0^{2\pi} \frac{e^{u(re^{i\theta})}}{r^m} \frac{r^{m+1}}{|f(re^{i\theta})|} d\theta < C,$$

we get $|\int_D \tilde{K} d\mu_{\tilde{g}}| < C$. Then $\int_D |\tilde{A}|^2 d\mu_{\tilde{f}} < +\infty$. Therefore, $\tilde{f}(\frac{1}{z})$ satisfies the conditions of Lemma 1.8.

Set $\hat{f}(z) = \tilde{f}(1/z)$, and $\hat{g} = d\hat{f} \otimes d\hat{f} = e^{2\hat{u}} g_{euc}$. We have

$$\hat{u}(z) = \tilde{u}(\frac{1}{z}) - 2\log|z| = m \log |\frac{1}{z}| - \log |f(\frac{1}{z})|^2 - 2 \log |z|.$$

Then

$$\lim_{z\to\infty} (\hat{u}(z) - m \log |z|) = -\lim_{z\to\infty} \log \frac{|f(\frac{1}{z})|^2}{|\frac{1}{z}|^{2m+2}} = -2\omega(0) + 2\log(m+1).$$

Applying Lemma 1.8 (2.3) and (2.1), we get $m = 0$.

□

We define

$$\mathcal{S}' = \{z \in \Sigma : \lim_{r\to 0} \varliminf_{k\to+\infty} \int_{B_r^{h_0}(z)} |A_{f_k}|^2 > \frac{\epsilon_0}{2}\}.$$

We need to prove $\mathcal{S}'$ is empty.

Assume $\mathcal{S}'$ is not empty. Given a point $p \in \mathcal{S}'$, we choose $U_k, U, \vartheta_k, \vartheta$ as in Theorem 3.1, and assume $p = 0$. We can choose U_k such that $U_k \cap \mathcal{S}' = \{p\}$. Let

$$\hat{f}_k = f_k \circ \vartheta_k$$

and note that $\hat{f}_k$ is a conformal map from D into $\mathbb{R}^3$. Let

$$g_{\hat{f}_k} = e^{2\hat{u}_k} g_{euc}, \quad h_k = e^{2v_k} g_{euc}.$$

Note that 0 is the only point in D which satisfies

$$\lim_{r\to 0} \varliminf_{k\to+\infty} \int_{D_r(z)} |A_{\hat{f}_k}| d\mu_{\hat{f}_k} > \frac{\epsilon_0}{2}.$$

Put

$$\int_{D_{r_k}(z_k)} |A_{f_k}|^2 = \frac{\epsilon_0}{2}, \quad and \quad \int_{D_r(z)} |A_{f_k}|^2 < \frac{\epsilon_0}{2}, \quad \forall D_r(z) \subset D_{\frac{1}{2}}, \quad r < r_k.$$

Then $z_k \to 0$ and $r_k \to 0$. Let $f'_k = \frac{\hat{f}_k(r_k z + z_k) - \hat{f}_k(z_k)}{\lambda_k}$, where

$$\lambda_k = diam(\hat{f}(z_k + [0, 1/2])).$$

By Theorem 4.1,

$$\|u'_k\|_{L^\infty(D_r)} \le C(r), \quad \forall r \in (0, 1).$$

Then, by Theorem 3.2, f'_k converges smoothly on $D_{\frac{3}{4}}$.

For any point $z_0 \in \partial D_{\frac{1}{2}}$, put

$$\gamma_k(t) = \frac{1}{4}(1+t)z_0, \quad t \in [0, 1], \quad and \quad \tau_k = diam(f'_k(\gamma_k)).$$

Then by Theorem 4.1 and Theorem 3.2, $\frac{f'_k - f'_k(z_0)}{\tau_k}$ converges smoothly. Since f'_k converges in $D_{\frac{3}{4}}$, we may assume $\tau_k \to \tau_0 > 0$. Then f'_k converges smoothly on $D_{\frac{3}{4}}(z_0)$. Thus f'_k converges smoothly on D_1. In this way, we can prove that a subsequence of f'_k converges smoothly on D_R for any R. Let f'_0 be the limit. Then $u'_0 \in L^\infty_{loc}(\mathbb{C})$ and

$$\int_D |A_{f'_0}|^2 = \frac{\epsilon_0}{2}.$$

Obviously, f'_0 is proper. If $diam(f'_0) = +\infty$, then f'_0 is noncompact and complete. Then by Theorem 1.3, f'_0 is a plane which implies that $\int_D |A_{f'_0}|^2 = 0$. A contradiction. So, $diam(f'_0) < +\infty$, then by Simon's inequality [18], $\mu(f'_0) < +\infty$. By Proposition 4.1 in [8], f'_0 can be considered as a continuous map from S^2 into $\mathbb{R}^3$.

Now, we set $\hat{f}'_k(z) = \hat{f}_k(z_k + z)$ and

$$\mathcal{S}(\hat{f}'_k) = \{z \in \mathbb{C} \setminus \{0\} : \lim_{r\to 0} \varliminf_{k\to+\infty} \int_{D_r(z)} |A_{\hat{f}'_k}|^2 > \frac{\epsilon_0}{2}\},$$

and

$$\Gamma(\theta_1, \theta_2, t) = \{te^{i\theta} : \theta_1 \le \theta \le \theta_2\}.$$

Since $\mathcal{S}(\hat{f}'_k)$ is a finite set, we can choose $\theta_1 < \theta_2$, such that

$$\left(\cup_{t\in[\frac{r_k}{r}, r]} \Gamma(\theta_1, \theta_2, t)\right) \cap \mathcal{S}(\hat{f}'_k) = \emptyset. \tag{3.1}$$

Take $t_k\in[\frac{r_k}{r},r]$, such that

$$\lambda'_k=diam(\hat{f}'_k(\Gamma(\theta_1,\theta_2,t_k)))=\inf_{t\in[\frac{r_k}{r},r]}diam(\hat{f}'_k(\Gamma(\theta_1,\theta_2,t))).$$

By Proposition 4.1 in [8],

$$\lim_{t\to 0}\lim_{k\to+\infty}diam(\hat{f}'_k(\Gamma(\theta_1,\theta_2,t)))=0,\quad \lim_{t\to\infty}\lim_{k\to+\infty}diam(f'_k(\Gamma(\theta_1,\theta_2,t)))=0.$$

Then

$$t_k\to 0,\quad and\quad \frac{t_k}{r_k}\to+\infty.$$

Let

$$f''_k=\frac{\hat{f}_k(t_kz+z_k)-\hat{f}_k(t_ke^{i\theta_1}+z_k)}{\lambda'_k}$$

and

$$\mathcal{S}(\{f''_k\})=\{z\in\mathbb{C}\setminus\{0\}:\lim_{r\to 0}\varliminf_{k\to+\infty}\int_{D_r(z)}|A_{\hat{f}_k}|^2>\frac{\epsilon_0}{2}\}.$$

By (3.1) and Theorem 3.2 and Theorem 4.1, f_k converges smoothly near $\Gamma(\theta_1,\theta_2,1)$. Following the method we get f'_0, we obtain that f''_k converges smoothly on any compact subset of $\mathbb{C}\setminus(\{\mathcal{S}(f''_k)\}\cup\{0\})$. Let f''_0 be the limit. Then

$$diam(f''_0(\Gamma(\theta_1,\theta_2,t)))=\inf_{t\in(0,\infty)}diam(f''_0(\Gamma(\theta_1,\theta_2,t))). \tag{3.2}$$

Then $\mu_{f''_0}(D_r(0))=\infty$ and $\mu_{f''_0}(\mathbb{C}\setminus D_r(0))=\infty$ for any r. Otherwise, by Proposition 4.1 in [8],

$$\lim_{t\to 0}diam(f''_0(\Gamma(\theta_1,\theta_2,t)))=0,\quad or\quad \lim_{t\to+\infty}diam(f''_0(\Gamma(\theta_1,\theta_2,t)))=0.$$

It contradicts (3.2). Thus f''_0 is complete, noncompact and has at least 2 ends.

Now, choose y_0 such that

$$d(y_0,\frac{f_k(\Sigma)-\hat{f}_k(t_ke^{i\theta_1}+z_k)}{\lambda'_k})>\delta>0$$

Set $I=\frac{y-y_0}{|y-y_0|^2}$. Then $I(f''_k)$ converges to $I(f''_0)$ smoothly on any compact subset of $\mathbb{C}\setminus(\{0\}\cup\mathcal{S}(\{f''_k\}))$. For any small r and any $z\in\mathcal{S}(\{f''_k\})\cup\{0\}$, since $I(f''_k)$ is Willmore on $D_r(z)$ and converges smoothly to $I(f''_0)$ on $\partial D_r(z)$, we get $Res(I(f''_0),z)=0$ (for the definition of Res, one can refer to [10]). Then, by Lemma 4.1 in [10] (see also Theorem I.6 in [16]) and Lemma 3.3, $I(f''_0)$ is a smooth Willmore embedding on $D_r(z)$. Moreover, for a large R, since $I(f''_k)$ is Willmore on D_R and converges smoothly on ∂D_R, $Res(I(f''_0),\infty)$ is also 0. Then $I(f''_0)(\frac{1}{z})$ is a smooth Willmore embedding on $D_{\frac{1}{R}}$.

Therefore, $I(f''_0)$ can be considered as a smooth conformal immersion from S^2 into $\mathbb{R}^3$. Obviously, $I(f''_0)$ has no transversal self-intersections. By Lemma 1.6, $I(f''_0)$ must be a round sphere. It contradicts the fact that f''_0 has at least 2 ends.

Hence we get $\mathcal{S}'=\emptyset$.

Then, using the argument in [8], we get $\|u_k\|_{L^\infty(\Sigma)}<C$ (this can also be deduced from Theorem 4.1). Given a point $p\in\Sigma$, we choose $U_k,U,\vartheta_k,\vartheta$ as in Theorem 3.1, and assume $p=0$. Let $\hat{f}_k=f_k\circ\vartheta_k$, which is conformal. Then we can choose an r, such

that $\int_{D_r}|A_{\hat{f}}|^2<\epsilon_0$. Using Theorem 3.2, $\hat{f}$ converges smoothly on $D_{\frac{r}{2}}$. We can choose r to be sufficiently small, such that their exists r_p, such that $B^{h_0}_{r_p}(p)\subset\varphi_k(D_{\frac{r}{2}})$. Thus f_k converges smoothly on $B^{h_0}_{r_p}(p)$. □

4. Appendix

The proof of the following theorem can be found in [12]. But for the convenience of the readers, we provide a proof in this appendix.

Theorem 4.1. *Let $f_k:D\to\mathbb{R}^n$ be a smooth conformal immersion which satisfies*

1) $\int_D|A_{f_k}|^2d\mu_{f_k}<\gamma_n-\tau$, *where $\tau>0$ and*

$$\gamma_n=\begin{cases}8\pi & when\ n=3\\ 4\pi & when\ n\geq 4.\end{cases}$$

2) $f_k(D)$ *can be extended to a closed immersed surface Σ_k with*

$$\int_{\Sigma_k}|A_{f_k}|^2d\mu_{f_k}<\Lambda.$$

Take a curve $\gamma:[0,1]\to D$, and set $\lambda_k=diam\,f_k(\gamma)$. Then we can find a subsequence of $f'_k=\frac{f_k-f_k(\gamma(0))}{\lambda_k}$ which converges weakly in $W^{2,2}_{loc}(D)$. Let $df'_k\otimes df'_k=e^{2u'_k}(dx^1\otimes dx^1+dx^2\otimes dx^2)$. For any $r<1$,

$$\|u'_k\|_{L^\infty(D_r)}<C(r).$$

Proof. Put $f'_k=\frac{f_k-f_k(\gamma(0))}{\lambda_k}$, $\Sigma'_k=\frac{\Sigma_k-f_k(\gamma(0))}{\lambda_k}$. We have two cases:

Case 1: $diam(f'_k)<C$. By inequality (1.3) in [18] with $\rho=\infty$, $\frac{\Sigma'_k\cap B_\sigma(\gamma(0))}{\sigma^2}\leq C$ for any $\sigma>0$. Hence we get $\mu(f'_k)<C$ by taking $\sigma=diam(f'_k)$. Then by Helein's convergence theorem [6, 8], f'_k converges weakly in $W^{2,2}_{loc}(D)$. Since $diam\,f'_k(\gamma)=1$, the weak limit is not trivial.

Case 2: $diam(f'_k)\to+\infty$. We take a point $y_0\in\mathbb{R}^n$ and a constant $\delta>0$, s.t.

$$B_\delta(y_0)\cap\Sigma'_k=\emptyset,\quad\forall k.$$

Let $I=\frac{y-y_0}{|y-y_0|^2}$, and

$$f''_k=I(f'_k),\quad\Sigma''_k=I(\Sigma'_k).$$

By conformal invariance of Willmore functional [3, 19], we have

$$\int_{\Sigma''_k}|A_{\Sigma''_k}|^2d\mu_{\Sigma''_k}=\int_{\Sigma_k}|A_{\Sigma_k}|^2d\mu_{\Sigma_k}<\Lambda.$$

Since $\Sigma''_k\subset B_{\frac{1}{\delta}}(0)$, also by (1.3) in [18], we get $\mu(f''_k)<C$. Let

$$\mathcal{S}(\{f''_k\}):=\{p\in D:\lim_{r\to0}\varliminf_{k\to+\infty}\int_{D(p)}|A_{f''_k}|^2d\mu_{f''_k}\geq\gamma_n\}.$$

Then f''_k converges weakly in $W^{2,2}_{loc}(D\setminus\mathcal{S}(f''_k))$.

Next, we prove that f_k'' does not converge to a point by assumption. If f_k'' converges to a point in $W^{2,2}_{loc}(D\setminus\mathcal{S}(f_k''))$, then the limit must be 0, for $diam\,(f_k')$ converges to $+\infty$. By the definition of f_k'', we can find a $\delta_0>0$, such that $f_k''(\gamma)\cap B_{\delta_0}(0)=\emptyset$. Thus for any $p\in\gamma([0,1])\setminus\mathcal{S}(f_k'')$, f_k'' will not converge to 0. A contradiction.

Then we only need to prove that f_k' converges weakly in $W^{2,2}_{loc}(D,\mathbb{R}^n)$. Let f_0'' be the limit of f_k'' which is a branched immersion of D in $\mathbb{R}^n$. Let $\mathcal{S}^*=f_0''^{-1}(\{0\})$, which is isolate. Note that for any $z_0\in\mathcal{S}^*$, there exits $m>0$, such that

$$\lim_{|z-z_0|\to 0}\frac{|f(z)-f(z_0)|}{|z-z_0|^m}>0.$$

First, we prove that for any $\Omega\subset\subset D\setminus(\mathcal{S}^*\cup\mathcal{S}(\{f_k''\}))$, f_k' converges weakly in $W^{2,2}(D,\mathbb{R}^n)$: Since f_0'' is continuous on $\bar{\Omega}$, we may assume $dist(0,f_0''(\Omega))>\delta>0$. Then $dist(0,f_k''(\Omega))>\frac{\delta}{2}$ when k is sufficiently large. Noting that $f_k'=\frac{f_k''}{|f_k''|^2}+y_0$, we get that f_k' converges weakly in $W^{2,2}(\Omega,\mathbb{R}^n)$.

Next, we prove that for each $p\in\mathcal{S}^*\cup\mathcal{S}(\{f_k''\})$, f_k' also converges in a neighborhood of p.

Let $g_{f_k'}=e^{2u_k'}g_{euc}$. Since $f_k'\in W^{2,2}_{conf}(D_{4r}(p))$ with $\int_{D_{4r}(p)}|A_{f_k'}|^2d\mu_{f_k'}<8\pi-\tau$ when r is sufficiently small and k sufficiently large, by the arguments in [8], we can find a v_k solving the equation

$$-\Delta v_k=K_{f_k'}e^{2u_k'},\quad z\in D_r\quad and\quad \|v_k\|_{L^\infty(D_r(p))}<C.$$

Since f_k' converges to a conformal immersion in $D_{4r}\setminus D_{\frac{1}{4}r}(p)$, we may assume that

$$\|u_k'\|_{L^\infty(D_{2r}\setminus D_r(p))}<C.$$

Then $u_k'-v_k$ is a harmonic function with $\|u_k'-v_k\|_{L^\infty(\partial D_{2r}(p))}<C$, then we get $\|u_k'(z)-v_k(z)\|_{L^\infty(D_{2r}(p))}<C$ from the Maximum Principle. Thus, $\|u_k'\|_{L^\infty(D_{2r}(p))}<C$, which implies $\|\nabla f_k'\|_{L^\infty(D_{2r})}<C$. By the equation $\Delta f_k'=e^{2u_k'}H_{f_k'}$, and the fact that

$$\|e^{2u_k'}H_{f_k'}\|^2_{L^2(D_{2r})}<e^{2\|u_k'\|_{L^\infty(D_{2r}(p))}}\int_{D_{2r}}|H_{f_k'}|^2d\mu_{f_k'},$$

we get $\|\nabla f_k'\|_{W^{1,2}(D_r)}<C$. We complete the proof.

□

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